

Radiation from elementary sources in a uniaxial wire medium

Mário G. Silveirinha* and Stanislav I. Maslovski†

*Departamento de Engenharia Electrotécnica
Instituto de Telecomunicações, Universidade de Coimbra
Pólo II, 3030-290 Coimbra, Portugal*

(Dated: February 22, 2012)

We investigate the radiation properties of two types of elementary sources embedded in a uniaxial wire medium: a short dipole parallel to the wires and a lumped voltage source connected across a gap in a generic metallic wire. It is demonstrated that the radiation pattern of these elementary sources have quite anomalous and unusual properties. Specifically, the radiation pattern of a short vertical dipole resembles that of an isotropic radiator close to the effective plasma frequency of the wire medium, whereas the radiation from the lumped voltage generator is characterized by an infinite directivity and a non-diffractive far-field distribution.

PACS numbers: 42.70.Qs, 78.20.Ci, 41.20.Jb

I. INTRODUCTION

Wire media, generically defined as structured materials formed by arrays of long metallic wires,^{1–3} is perhaps the class of metamaterials whose effective response is better understood. Particularly, during the last decade a vast body of theoretical methods and analytical tools have been developed which enable characterizing the effective electromagnetic response of wire-based materials in different scenarios with great accuracy.^{4–18} However, a bit surprisingly, the problem of radiation by localized external sources embedded within wire media, has only been cursorily discussed in the literature.^{19–22}

This gap can be explained in part because of the peculiar electromagnetic response of wire media, which are typically characterized by strong spatial dispersion in the long wavelength limit,⁶ and this property greatly complicates the analytical modeling. In simple terms, a medium is spatially dispersive if the polarization vector at some generic point in space depends not only on the macroscopic electric field, but also on the gradient of the field, and possibly higher order derivatives.²³

The objective of this work is to characterize the radiation properties of elementary external localized sources placed within a wire medium using an effective medium approach. Specifically, we are interested in the following two scenarios: (i) a short vertical dipole is embedded in the wire medium (Fig. 1a) and (ii) an external lumped voltage source is connected across a gap in a generic metallic wire (Fig. 1b). As will be detailed in Section II, these sources are modeled in terms of the Dirac delta function (Dirac- δ). It is, however, important to make clear at the outset that an effective medium description of the radiation problem is only possible if the source is localized on a larger scale than the characteristic dimension of the metamaterial (e.g., the lattice constant a , see Fig. 1). Hence, the short vertical dipole considered here should be understood as some external current spread over a region of space whose characteristic diameter in the xy -plane is larger or equal than a , but much smaller than the wavelength. Similarly, even though for the purpose of illustration and discussion we say that in

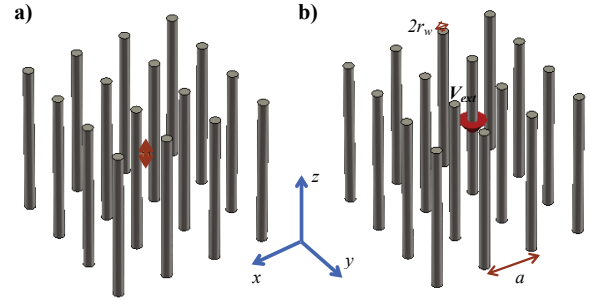


FIG. 1: (Color online) Uniaxial wire medium formed by a square lattice of metallic wires oriented along the z -direction. (a) Excitation based on a short vertical dipole embedded in the wire medium. (b) Excitation based on a discrete voltage source connected directly at the center of one of the wires.

case (ii) the voltage source is connected across the gap of a single wire, it is more accurate to imagine such source as an array of voltage generators, distributed over a region of space whose characteristic diameter is larger than a , being each voltage generator connected across a gap in a metallic wire lying within the mentioned region. With the exception of the immediate vicinity of these sources, the solution determined with our theory (based on the Dirac- δ distribution) should describe accurately the radiated fields.

One of the challenges in the characterization of the radiation by a localized source within a wire medium is related to the calculation of quantities such as the Poynting vector or the radiation intensity (i.e. the power radiated per unit of solid angle). Indeed, in general the usual form of the Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, does not hold in case of spatially dispersive materials.^{23,24} Moreover, there is no known theory to determine the Poynting vector in a general spatially dispersive material, and the only case that is actually understood and for which closed analytical formulas are available, is when the electromagnetic fields have a plane wave type-spatial variation.²³ In this work, we derive closed analytical formulas that enable calculating explicitly the Poynting vector and the elec-

tromagnetic energy density in uniaxial wire media for *arbitrary* electromagnetic field distributions. This is one of the key results of the paper.

To do this, we rely on the theory of our earlier works,^{17,25} where we have shown that the effective medium response of the wire medium can be modeled using a quasi-static model, based on the introduction of two additional variables I and φ_w . What is remarkable about such a model, is that the relations between the macroscopic electromagnetic fields and the additional variables I and φ_w are *local* in space. Therefore, such formalism enables describing the unconventional electrodynamics of the wire medium using a local approach, without requiring the definition of an effective spatially dispersive dielectric function, which would lead to nonlocal relations between the polarization vector and the macroscopic electric field.⁶ It is important to mention that the introduction of the additional variables is not just a trick that simplifies the modeling of the wire medium: it is actually full of physical significance, and elucidates about the internal physical processes that determine the macroscopic response of the metamaterial. Indeed, the variable I can be understood as the electric current that flows along the metallic wires (interpolated in such a manner that it becomes a continuous function defined in all space), whereas the variable φ_w can be understood as the average potential drop from a given wire to the boundary of the respective unit cell (the potential is interpolated in the same manner as the current). For more details, the reader is referred to Refs. 17,25.

This paper is organized as follows. In Section II, we briefly review the quasi-static model of the wire medium, and formulate the radiation problem for the two excitations of interest. In Section III we solve the pertinent radiation problem in the spectral domain. First, we discuss the general case of a stratified (along z) structure, and after this we analyze in details the particular case of an unbounded uniform structure. In Section IV, we show that for an unbounded uniform structure the fields radiated by the elementary external sources can be as well directly determined from the nonlocal dielectric function of the metamaterial. After this, in Section V we derive a general Poynting theorem that expresses the conservation of energy in wire media, and in Section VI we use these results to obtain the asymptotic form of the Poynting vector in the far-field, as well as the directive gain, directivity, and the power radiated by a short vertical dipole. The conclusions are drawn in Section VII. In this work, we assume that in case of time harmonic regime the time variation is of the form $e^{j\omega t}$.

II. LOCAL FORMULATION BASED ON THE INTRODUCTION OF ADDITIONAL VARIABLES

In Refs. 17,25 it was shown that the internal physical processes that determine the macroscopic response of a wide class of wire media, are intrinsically related to the dynamics of the electric current I along the wires

and the additional potential φ_w , whose physical meaning was already discussed in Introduction. In particular, it was proven that for the case of straight wires oriented along the z -direction the macroscopic electromagnetic fields satisfy

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \quad (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{ext} + \frac{I}{A_c}\hat{\mathbf{z}} + j\omega\varepsilon_h\mathbf{E}, \quad (2)$$

where ε_h is the permittivity of the host material, $A_c = a^2$ is the area of the unit cell, and a is the period of the wire medium (Fig. 1). Notice that unlike in our previous works,^{17,25} here we admit the possibility of an external distributed current source \mathbf{J}_{ext} . The electromagnetic fields are coupled to the current I and additional potential φ_w via a set of transmission line-type equations:

$$\frac{\partial\varphi_w}{\partial z} = -(Z_w + j\omega L)I + E_z + V_{ext}A_c\delta(x, y, z), \quad (3)$$

$$\frac{\partial I}{\partial z} = -j\omega C\varphi_w. \quad (4)$$

In the above, C , L , and Z_w represent the capacitance, inductance, and self-impedance of a wire per unit of length, respectively, and explicit formulas for those parameters can be found in our previous papers. The real part of Z_w is related to the ohmic loss in the metallic wires, whereas its imaginary part is related to the kinetic inductance of the electrons in the metal. As compared to Refs. 17,25 now we allow for an external lumped voltage source (with amplitude V_{ext}) to be placed across a gap in the wire in the central unit cell. It is simple to check based on the theory of Ref. 17, that this lumped voltage source is modeled by the term $V_{ext}A_c\delta(x, y, z)$. Notice that similar to the current and additional potential, the lumped generator is interpolated so that it becomes a function defined over all space.

In the next section, we will determine the solution of the radiation problems sketched in Fig. 1, based on the system of Eqs. (1)–(4). It should be emphasized that such a system of equations is *local* in the sense that all the relevant medium parameters (C , L , Z_w , ε_h , and μ_0) are independent of the gradient and higher order derivatives of the electromagnetic fields, I , and φ_w . This contrasts with the usual formulation based on the effective dielectric function, which does not require the introduction of additional variables but in which the dielectric function depends explicitly on the wave vector.^{6,12} This will be further discussed in Section IV.

For future reference, we note that from Eqs. (3)–(4) it follows that

$$\begin{aligned} \frac{\partial}{\partial z} \frac{1}{C} \frac{\partial I}{\partial z} + (\omega^2 L - j\omega Z_w) I \\ = -j\omega [E_z + V_{ext}A_c\delta(x, y, z)]. \end{aligned} \quad (5)$$

In the above, it was supposed that L , C , and Z_w may depend on z (but not on x and y), which can happen in case of a stratified wire medium (with direction of stratification along z), such that either the permittivity of the host medium or the radii of the wires varies with z .

III. THE RADIATION PROBLEM

Next, we derive the solution of the radiation problem in terms of two scalar potentials. We admit that the external current density describes a short vertical dipole, so that that $\mathbf{J}_{ext} = j\omega p_e \delta(x, y, z) \hat{\mathbf{z}}$, where p_e represents the electric dipole moment. Since the Maxwell equations are linear it is possible to solve the two radiation problems sketched in Fig. 1 simultaneously. This will be done in what follows.

A. Solution in terms of two scalar potentials for the general case of a stratified structure

For generality, in this subsection, we admit that L , C , Z_w , and ε_h may depend on z , which as discussed previously, may be useful to study problems of radiation in stratified media. We look for a solution of (1)–(4) such that the macroscopic electromagnetic fields are written in terms of a Hertzian potential Φ so that

$$\mathbf{H} = \nabla \times \{j\omega\Phi\hat{\mathbf{z}}\}, \quad (6)$$

$$\mathbf{E} = \omega^2\mu_0\Phi\hat{\mathbf{z}} + \nabla\left(\frac{1}{\varepsilon_h}\frac{\partial\Phi}{\partial z}\right). \quad (7)$$

It can be easily verified that (6)–(7) satisfy, indeed, the Maxwell equations (1)–(2), provided that

$$\begin{aligned} \varepsilon_h \frac{\partial}{\partial z} \frac{1}{\varepsilon_h} \frac{\partial\Phi}{\partial z} + \nabla_t^2\Phi + \omega^2\mu_0\varepsilon_h\Phi + \frac{I}{j\omega A_c} \\ = -p_e\delta(x, y, z), \end{aligned} \quad (8)$$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Hence, substituting (7) into (5) and using the above result, it follows that

$$\begin{aligned} C \frac{\partial}{\partial z} \frac{1}{C} \frac{\partial}{\partial z} \left(\frac{I}{j\omega A_c} \right) + (\omega^2 LC - j\omega Z_w C) \left(\frac{I}{j\omega A_c} \right) \\ = -\frac{C}{A_c \varepsilon_h} [\varepsilon_h E_z + \varepsilon_h V_{ext} A_c \delta(x, y, z)] \\ = -\frac{C}{A_c \varepsilon_h} \left[-\nabla_t^2\Phi - p_{ef}\delta(x, y, z) - \frac{I}{j\omega A_c} \right], \end{aligned} \quad (9)$$

where we defined the effective dipole moment for the combined excitations:

$$p_{ef} = p_e - \varepsilon_h A_c V_{ext}. \quad (10)$$

Notice that p_{ef} depends on both p_e and V_{ext} , because we allow for the simultaneous excitation of the wire medium with the two pertinent types of elementary sources.

For convenience, let us introduce the auxiliary potential

$$\psi = \frac{1}{k_p^2} \frac{I}{j\omega A_c}, \quad (11)$$

where $k_p = \sqrt{\mu_0/(LA_c)}$ is the so-called plasma wave number of the wire medium,^{6,12,17} which may be calculated using, for example, the approximate formula

applicable to both square and hexagonal wire lattices $k_p \approx (1/a)\sqrt{2\pi/\log[a^2/4r_w(a-r_w)]}$, where r_w is the radius of the metallic wires. Using the fact that for straight unloaded wires $LC = \mu_0\varepsilon_h$, it follows that Eqs. (8) and (9) are equivalent to

$$\varepsilon_h \frac{\partial}{\partial z} \frac{1}{\varepsilon_h} \frac{\partial\Phi}{\partial z} + \nabla_t^2\Phi + k_h^2\Phi + k_p^2\psi = -p_e\delta(x, y, z), \quad (12)$$

$$C \frac{\partial}{\partial z} \frac{1}{C} \frac{\partial\psi}{\partial z} + (k_h^2 + \beta_c^2 - k_p^2)\psi - \nabla_t^2\Phi = p_{ef}\delta(x, y, z), \quad (13)$$

where we put $k_h^2 = \omega^2\mu_0\varepsilon_h$ and $\beta_c^2 = -j\omega Z_w C$. Hence, to determine the solution of our problem, we need to solve this coupled system of partial differential equations with unknowns Φ and ψ .

To do this, it is most convenient to work in the Fourier domain. Defining $\tilde{\Phi}$ and $\tilde{\psi}$ as the Fourier transform of Φ and ψ in the xy plane, respectively, so that

$$\tilde{\Phi} = \int \int \Phi e^{j(k_x x + k_y y)} dx dy, \quad (14)$$

and $\tilde{\psi}$ is defined similarly, it follows that

$$\varepsilon_h \frac{\partial}{\partial z} \frac{1}{\varepsilon_h} \frac{\partial\tilde{\Phi}}{\partial z} + (k_h^2 - k_t^2)\tilde{\Phi} + k_p^2\tilde{\psi} = -p_e\delta(z), \quad (15)$$

$$C \frac{\partial}{\partial z} \frac{1}{C} \frac{\partial\tilde{\psi}}{\partial z} + (k_h^2 + \beta_c^2 - k_p^2)\tilde{\psi} + k_t^2\tilde{\Phi} = p_{ef}\delta(z), \quad (16)$$

where $k_t^2 = k_x^2 + k_y^2$. Thus, we have reduced the radiation problem to the solution of a system of linear ordinary differential equations.

B. The case of a homogeneous medium

From hereafter, we restrict our attention to the particular case of a homogeneous and uniform medium, for which the structural parameters ε_h , C , L , and Z_w can be assumed independent of z . In such a case, the system (15)–(16) can be rewritten in a compact matrix notation as follows:

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} k_t^2 - k_h^2 & -k_p^2 \\ -k_t^2 & k_p^2 - k_h^2 - \beta_c^2 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} \\ + \delta(z) \begin{pmatrix} -p_e \\ p_{ef} \end{pmatrix}. \end{aligned} \quad (17)$$

The general solution of the homogeneous problem, when $p_e = p_{ef} = 0$, can be easily found using standard methods, and is given by

$$\begin{aligned} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = (C_1^+ e^{-\gamma_{TM} z} + C_1^- e^{+\gamma_{TM} z}) \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{TM}^2 \end{pmatrix} \\ + (C_2^+ e^{-\gamma_{qT} z} + C_2^- e^{+\gamma_{qT} z}) \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{qT}^2 \end{pmatrix}. \end{aligned} \quad (18)$$

where $\gamma_h^2 = k_t^2 - k_h^2$, C_i^\pm with $i = 1, 2$ are integration constants, and γ_{qT} and γ_{TM} are the propagation constants

along the z direction of the so-called quasi-transverse electromagnetic (qT) and transverse magnetic (TM) modes supported by the bulk wire medium. These parameters are defined consistently with Refs. 12,26, and satisfy

$$\gamma_{TM} = j \left[k_h^2 - \frac{1}{2} (k_p^2 + k_t^2 - \beta_c^2 + \sqrt{(k_p^2 + k_t^2 - \beta_c^2)^2 + 4k_t^2 \beta_c^2}) \right]^{\frac{1}{2}}, \quad (19)$$

$$\gamma_{qT} = j \left[k_h^2 - \frac{1}{2} (k_p^2 + k_t^2 - \beta_c^2 - \sqrt{(k_p^2 + k_t^2 - \beta_c^2)^2 + 4k_t^2 \beta_c^2}) \right]^{\frac{1}{2}}. \quad (20)$$

In case of perfectly conducting wires, we have $Z_w = 0$, and thus $\beta_c = 0$. In such a case the propagation constants of the qT and TM modes reduce to the well known forms, $\gamma_{qT} = jk_h$ and $\gamma_{TM} = \sqrt{k_p^2 + k_t^2 - k_h^2}$, respectively.⁶

Since the solution of (17) is obviously an even function of z , we may try a solution of the form

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = C_{qT} e^{-\gamma_{qT}|z|} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{qT}^2 \end{pmatrix} + C_{TM} e^{-\gamma_{TM}|z|} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{TM}^2 \end{pmatrix}. \quad (21)$$

By direct substitution into (17), it is readily found that the unknown constants C_{qT} and C_{TM} are required to satisfy

$$\begin{aligned} \gamma_{qT} C_{qT} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{qT}^2 \end{pmatrix} + \gamma_{TM} C_{TM} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{TM}^2 \end{pmatrix} \\ = -\frac{1}{2} \begin{pmatrix} -p_e \\ p_{ef} \end{pmatrix}. \end{aligned} \quad (22)$$

This yields

$$C_{qT} = \frac{1}{2\gamma_{qT}} \frac{(\gamma_h^2 - \gamma_{TM}^2) p_e + k_p^2 p_{ef}}{\gamma_{qT}^2 - \gamma_{TM}^2} \frac{1}{k_p^2}, \quad (23)$$

$$C_{TM} = \frac{1}{2\gamma_{TM}} \frac{(\gamma_h^2 - \gamma_{qT}^2) p_e + k_p^2 p_{ef}}{\gamma_{TM}^2 - \gamma_{qT}^2} \frac{1}{k_p^2}. \quad (24)$$

Substituting this result into Eq. (21), we finally obtain the desired solution

$$\begin{aligned} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = \frac{1}{2\gamma_{qT}} \frac{(\gamma_h^2 - \gamma_{TM}^2) p_e + k_p^2 p_{ef}}{\gamma_{qT}^2 - \gamma_{TM}^2} e^{-\gamma_{qT}|z|} \begin{pmatrix} 1 \\ \frac{\gamma_h^2 - \gamma_{qT}^2}{k_p^2} \end{pmatrix} + \\ \frac{1}{2\gamma_{TM}} \frac{(\gamma_h^2 - \gamma_{qT}^2) p_e + k_p^2 p_{ef}}{\gamma_{TM}^2 - \gamma_{qT}^2} e^{-\gamma_{TM}|z|} \begin{pmatrix} 1 \\ \frac{\gamma_h^2 - \gamma_{TM}^2}{k_p^2} \end{pmatrix}. \end{aligned} \quad (25)$$

The inverse Fourier transform of $\tilde{\Phi}$ is given by

$$\begin{aligned} \Phi &= \frac{1}{(2\pi)^2} \int \int \tilde{\Phi} e^{-j(k_x x + k_y y)} dk_x dk_y \\ &= \frac{1}{2\pi} \int_0^{+\infty} \tilde{\Phi} J_0(k_t \rho) k_t dk_t, \end{aligned} \quad (26)$$

where J_0 is the zero-order Bessel function of the first kind, $\rho = \sqrt{x^2 + y^2}$, and in the second identity we used the fact that $\tilde{\Phi}$ is a function of k_t . In general, this Sommerfeld-type integral can only be evaluated using numerical methods. Obviously, it is possible to write a similar formula for ψ .

C. Perfectly electric conducting wires

Let us now study what happens when to a first approximation the metal can be modeled as a perfect electric conductor (PEC), so that $Z_w \approx 0$. In such a situation, Eq. (25) simplifies to

$$\begin{aligned} \begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} &= \frac{1}{2\gamma_{qT}} k_p^2 \frac{p_e - p_{ef}}{k_p^2 + k_t^2} e^{-\gamma_{qT}|z|} \begin{pmatrix} 1 \\ \frac{k_t^2}{k_p^2} \end{pmatrix} \\ &+ \frac{1}{2\gamma_{TM}} \frac{k_t^2 p_e + k_p^2 p_{ef}}{k_p^2 + k_t^2} e^{-\gamma_{TM}|z|} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned} \quad (27)$$

We will discuss separately the two scenarios of interest. Let us consider first that $V_{ext} = 0$, so that the metamaterial is excited solely with the short vertical dipole. In

this case, $p_{ef} = p_e$, and thus we obtain simply

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = p_e \frac{1}{2\gamma_{TM}} e^{-\gamma_{TM}|z|} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (28)$$

The corresponding inverse Fourier transforms can be evaluated analytically in a trivial manner. This yields

$$\begin{pmatrix} \Phi \\ \psi \end{pmatrix} = p_e \frac{1}{4\pi r} e^{-jk_{ef}r} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (29)$$

with $k_{ef} = \sqrt{k_h^2 - k_p^2}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Hence, it is immediately seen that the qT mode (which in the case of a lossless metal is exactly transverse electromagnetic mode (TEM) with respect to the z -direction⁶) does not contribute to the radiation field of the short vertical dipole. This may look surprising at first, but it is actually simple to understand. Indeed, it is well known that the electric Green dyadic in a periodic structure (e.g. a photonic crystal or a metamaterial) can be written as a weighted summation of terms such as $\mathbf{E}_n \otimes \mathbf{E}_n$ where \mathbf{E}_n stands for a generic natural mode of the system, and \otimes represents the tensor product.²⁷ In particular, this immediately implies that a TEM mode (with respect to the z -direction) cannot possibly contribute to the field radiated by a short vertical dipole, because its contribution would be proportional to $\mathbf{E}_{TEM}(\mathbf{E}_{TEM} \cdot \hat{\mathbf{z}})$, whereas for a TEM mode $\mathbf{E}_{TEM} \cdot \hat{\mathbf{z}} = 0$. Notice that this discussion applies actually to the microscopic electromagnetic fields (before homogenization on the scale of the lattice constant), but it clearly indicates that the TEM mode cannot contribute as well to the radiated field in the framework of a macroscopic theory, consistent with Eq. (29). It is interesting to note that in the presence of loss the contribution of the qT mode to the radiation field does not vanish [the first addend of Eq. (25) does not vanish when $p_e = p_{ef}$], which is fully consistent with the microscopic theory, because in case of loss the electric field associated with the qT mode has a small longitudinal component (i.e. a component along the z direction).

The result (29) implies two unexpected things. First, despite the anisotropy of the wire medium, the wavefronts are spherical surfaces! Second, the emission of radiation is possible only above the effective plasma frequency of the metamaterial, $\omega_p = k_p/\sqrt{\epsilon_h\mu_0}$. The latter property is actually a direct implication of the TEM mode not being excited, as discussed above. To illustrate variation in space of the potential Φ , we plot in Fig. 2 the contour plots of Φ for two different frequencies of operation. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum. The plasma wave number of the effective medium is $k_p = 1.38/a$. Thus, the example of Fig. 2a corresponds to a frequency below ω_p , whereas the example of Fig. 2b corresponds to a frequency above ω_p . This explains that in the former case the potential is strongly localized in the vicinity of the dipole, whereas in the latter case the potential decays much more slowly as $1/r$.

At first glance, the result (29) could suggest that the electric field radiation pattern should be similar to that

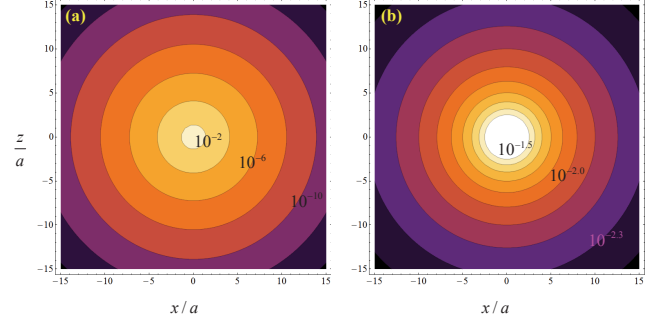


FIG. 2: (Color online) Contour plots of the amplitude of the potential Φ (arbitrary logarithmic unities). (a) $\omega a/c = 0.5$. (b) $\omega a/c = 1.5$. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum, and is excited by a short vertical dipole.

of a Hertzian dipole standing in a homogeneous isotropic plasma with a Drude dispersion. As it will be shown in Section VI, this is not true.

Next, we consider the case $p_e = 0$, so that the wire medium is excited by a lumped voltage source (Fig. 1b). In this scenario, the contribution of the TEM mode to the radiation field does not vanish. Indeed, the inverse Fourier transform of the first term in the right-hand side of (27) can be readily calculated and is equal to

$$\Phi_{qT} = -p_{ef} \frac{1}{2\gamma_{qT}} e^{-\gamma_{qT}|z|} \frac{k_p^2}{2\pi} K_0(k_p \rho), \quad (30)$$

where K_0 is the modified Bessel function of the second kind. On the other hand, the auxiliary potential (11) associated with the wire current satisfies $\psi_{qT} = -\frac{1}{k_p^2} \nabla_t^2 \Phi_{qT}$.

The result (30) is quite remarkable, because it predicts that the Hertzian potential, and hence the electromagnetic fields, varies with z simply as $e^{-\gamma_{qT}|z|} = e^{-jk_h|z|}$, and hence the radiated field is simply guided along z , without any form of decay. Moreover, Φ_{qT} is strongly localized in the vicinity of the z -axis, within a spatial region whose characteristic diameter is determined by $\lambda_p = 2\pi/k_p$. It should be mentioned that Φ_{qT} is actually singular over the z -axis (it has a logarithmic singularity). Such a singularity occurs because of the adopted δ -function model for the lumped voltage generator (also because the macroscopic model effectively assumes infinitesimal wire separation: $a \rightarrow 0$). The singularity disappears if one considers a less localized model for the discrete source, e.g. if $\delta(x, y, z)$ is replaced by $g(\rho)\delta(z)$, where g is some function of ρ concentrated near the origin. Even for such a source, the electromagnetic fields are characterized by a non-diffractive pattern. This is explained by the “canalization” properties of the wire medium, which enable the transport of the near-field with no diffraction.^{26,28} As far as we could check, the inverse Fourier transform of the second addend of (27), i.e. the contribution of the TM mode when $p_e = 0$, cannot be written in terms of the standard special functions, and hence it needs to be calculated numerically using (26).

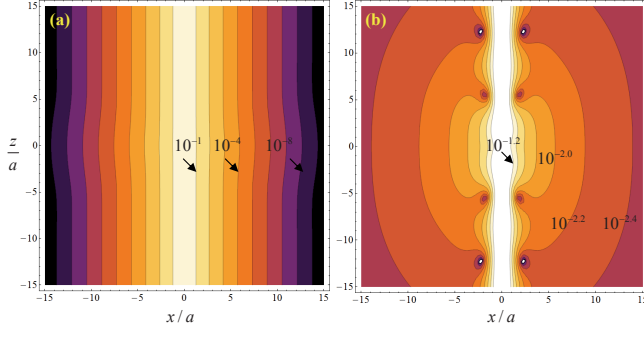


FIG. 3: (Color online) Contour plots of the amplitude of the potential Φ (arbitrary logarithmic unities). (a) $\omega a/c = 0.5$. (b) $\omega a/c = 1.5$. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum, and is excited by a lumped voltage generator. In case (b) an interference pattern of the TEM and TM modes is observed.

In Fig. 3, we plot the contour plots of Φ , for the same example as in Fig. 2. It is seen that when $\omega a/c = 0.5$ (Fig. 3a), i.e. below the effective plasma frequency, the fields are strongly concentrated close to the z -axis, and are guided away from the source with no diffraction. The contribution from the TM mode appears to be residual. On the other hand, above the plasma frequency (Fig. 3b), there are clearly two distinct emission channels, one associated with the TEM mode and another with the TM mode.

IV. NONLOCAL DIELECTRIC FUNCTION APPROACH

The objective of this section is to prove that the radiation problem in an unbounded uniform structure can be as well solved using the standard nonlocal dielectric function formalism.^{6,12,17} The case of a stratified structure, which can be easily handled with the theory of Section III, is out of reach of the nonlocal framework (it could however be handled with a combination of mode matching and additional boundary conditions,^{11,25} but a detailed discussion of it is out of the scope of this paper).

In a spatially dispersive medium, the Maxwell equations may be written in a compact form in the space domain as follows:

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \quad (31)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{ext} + j\omega\bar{\bar{\epsilon}}(\omega, j\nabla) \cdot \mathbf{E}, \quad (32)$$

where the dyadic operator $\bar{\bar{\epsilon}}(\omega, j\nabla)$ represents the effective dielectric function of the material. Notice that in the space domain the effective dielectric function should be regarded as a function of the gradient ∇ . This contrasts with the formulation of Section II where all the structural parameters are independent of ∇ . It is also possible to write the term $\bar{\bar{\epsilon}}(\omega, j\nabla) \cdot \mathbf{E}$ as a spatial convolution.²³ In the spectral (Fourier) domain, in which $j\nabla \leftrightarrow \mathbf{k}$, in the

particular case of the uniaxial wire medium formed by straight wires, the effective dielectric function is^{6,12,17}

$$\frac{\bar{\bar{\epsilon}}(\omega, \mathbf{k})}{\epsilon_h} = \bar{\bar{I}} - \frac{k_p^2 \hat{\mathbf{z}}\hat{\mathbf{z}}}{k_h^2 - j\xi k_h - k_z^2}. \quad (33)$$

where $\xi = (Z_w/L)\sqrt{\epsilon_h\mu_0}$, and the rest of the symbols are defined as in Section II. Notice that the effective dielectric function depends explicitly on $k_z \leftrightarrow j\frac{\partial}{\partial z}$.

Despite the apparently complicated form of Eqs. (31)–(32), the radiation problem can be readily solved in the spectral domain in case of an unbounded uniform structure. Indeed, by calculating the Fourier transform of both sides of the equations (31)–(32) with respect to all the space coordinates, so that $j\nabla \leftrightarrow \mathbf{k}$, it is readily found that

$$\mathbf{k} \times \mathbf{E}(\omega, \mathbf{k}) = \omega\mu_0\mathbf{H}(\omega, \mathbf{k}), \quad (34)$$

$$\mathbf{k} \times \mathbf{H}(\omega, \mathbf{k}) = -\omega\bar{\bar{\epsilon}}(\omega, \mathbf{k}) \cdot \mathbf{E}(\omega, \mathbf{k}) - \omega\mathbf{P}_{ext}(\omega, \mathbf{k}), \quad (35)$$

where $j\omega\mathbf{P}_{ext}(\omega, \mathbf{k}) = \mathbf{J}_{ext}(\omega, \mathbf{k})$ is the Fourier-transformed source term. After some straightforward manipulations, we find that the Fourier transform of the electric field is

$$\mathbf{E} = j\omega\mu_0 \left[\omega^2\mu_0\bar{\bar{\epsilon}}(\omega, \mathbf{k}) + \mathbf{k}\mathbf{k} - k^2\bar{\bar{I}} \right]^{-1} \cdot \mathbf{J}_{ext}, \quad (36)$$

and hence the electric field in the space domain can be formally written as

$$\mathbf{E}(\mathbf{r}) = \frac{j\omega\mu_0}{(2\pi)^3} \times \int \left[\omega^2\mu_0\bar{\bar{\epsilon}}(\omega, \mathbf{k}) + \mathbf{k}\mathbf{k} - k^2\bar{\bar{I}} \right]^{-1} \cdot \mathbf{J}_{ext}(\mathbf{k}) e^{-j\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k}. \quad (37)$$

Notice that, at least *a priori*, in the nonlocal dielectric function framework we can only consider excitations based on an external density of current (Fig. 1a). The characterization of the excitation based on a lumped voltage source requires the knowledge of internal degrees of freedom of the wire medium (e.g., the current along the wires and the additional potential), which are not described by the effective medium model. Nevertheless, ahead we will show that a lumped source V_{ext} can also be modeled by a suitable equivalent \mathbf{J}_{ext} .

Next, we obtain the solution of the radiation problem when $\mathbf{J}_{ext}(\mathbf{r}) = j\omega p_e \hat{\mathbf{z}}\delta(\mathbf{r})$, or equivalently when $\mathbf{P}_{ext}(\omega, \mathbf{k}) = p_e \hat{\mathbf{z}}$. Instead of attempting to calculate the integral (37) directly, we will solve instead (34)–(35) by introducing the Hertz potential Π_e . In this manner, we write the Fourier-transformed fields as follows:

$$\mathbf{E} = \omega^2\epsilon_h\mu_0\Pi_e - \mathbf{k}(\mathbf{k} \cdot \Pi_e), \quad (38)$$

$$\mathbf{H} = \omega\epsilon_h\mathbf{k} \times \Pi_e. \quad (39)$$

This form immediately satisfies (34). From (35) we find

$$\mathbf{k} \times (\mathbf{k} \times \Pi_e) = -\frac{\bar{\bar{\epsilon}}}{\epsilon_h} \cdot [k_h^2\Pi_e - \mathbf{k}(\mathbf{k} \cdot \Pi_e)] - \frac{p_e\hat{\mathbf{z}}}{\epsilon_h}. \quad (40)$$

Using (33), after some trivial vector algebra, we obtain

$$(k_h^2 - k^2)\mathbf{\Pi}_e = -\chi_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} \cdot [k_h^2\mathbf{\Pi}_e - \mathbf{k}(\mathbf{k} \cdot \mathbf{\Pi}_e)] - \frac{p_e\hat{\mathbf{z}}}{\varepsilon_h}, \quad (41)$$

where

$$\chi_{zz} = -\frac{k_p^2}{k_h^2 - j\xi k_h - k_z^2}. \quad (42)$$

Calculating the vector product of (41) by $\hat{\mathbf{z}}$ we find that

$$\hat{\mathbf{z}} \times \mathbf{\Pi}_e = 0, \quad (43)$$

and therefore, $\mathbf{\Pi}_e = \Pi_z\hat{\mathbf{z}}$. From this and (41),

$$[k_h^2 - k^2 + \chi_{zz}(k_h^2 - k_z^2)]\Pi_z = -\frac{p_e}{\varepsilon_h}. \quad (44)$$

For PEC wires $\xi = 0$, and thus from (42) we have

$$(k_h^2 - k_p^2 - k^2)\Pi_z = -\frac{p_e}{\varepsilon_h}, \quad (45)$$

from which

$$\Pi_z(\omega, \mathbf{k}) = -\frac{p_e}{\varepsilon_h(k_h^2 - k_p^2 - k^2)}, \quad (46)$$

and, thus,

$$\Pi_z(\omega, \mathbf{r}) = \frac{p_e}{\varepsilon_h} \frac{e^{-j\sqrt{k_h^2 - k_p^2}r}}{4\pi r}, \quad (47)$$

which is the same as (29), because $\Phi = \varepsilon_h\Pi_z$.

In the general case in which the metal has a plasmonic-type response, $\xi \neq 0$. Introducing the notation $\beta_c^2 = -j\xi k_h$ we obtain from (44)

$$\begin{aligned} \Pi_z(\omega, \mathbf{k}) &= -\frac{p_e}{\varepsilon_h} \frac{k_h^2 + \beta_c^2 - k_z^2}{(k_h^2 - k^2)(k_h^2 + \beta_c^2 - k_z^2) - k_p^2(k_h^2 - k_z^2)} \\ &= -\frac{p_e}{\varepsilon_h} \frac{k_h^2 + \beta_c^2 - k_z^2}{(k_z^2 + \gamma_{TM}^2)(k_z^2 + \gamma_{qT}^2)}, \end{aligned} \quad (48)$$

where γ_{TM} and γ_{qT} are given by (19) and (20).

Calculating the inverse Fourier transform with respect to k_z we find

$$\begin{aligned} \Pi_z(\omega, \mathbf{k}_t, z) &= \frac{p_e}{2\varepsilon_h} \left(\frac{k_h^2 + \beta_c^2 + \gamma_{qT}^2}{\gamma_{qT}(\gamma_{qT}^2 - \gamma_{TM}^2)} e^{-\gamma_{qT}|z|} \right. \\ &\quad \left. + \frac{k_h^2 + \beta_c^2 + \gamma_{TM}^2}{\gamma_{TM}(\gamma_{TM}^2 - \gamma_{qT}^2)} e^{-\gamma_{TM}|z|} \right). \end{aligned} \quad (49)$$

At first glance this result looks different from (25), but one may verify that $k_h^2 + \beta_c^2 + \gamma_{qT}^2 = (\gamma_h^2 - \gamma_{TM}^2) + k_p^2$, and, similarly, $k_h^2 + \beta_c^2 + \gamma_{TM}^2 = (\gamma_h^2 - \gamma_{qT}^2) + k_p^2$. Thus, we recover (25) with $p_e = p_{ef}$, which corresponds to the

case $V_{ext} = 0$, consistent with our assumptions in the beginning of this section.

Surprisingly, the lumped voltage source V_{ext} in (3)–(4) can be equivalently represented *within the nonlocal dielectric function model* with some distributed current density $\mathbf{J}_{ext,V}$ in the unbounded wire medium. To show this, we consider the Fourier-transformed equations (3)–(4) (for simplicity we let $Z_w = 0$), from which the Fourier-transformed current $I(\omega, \mathbf{k})$ can be expressed as

$$I(\omega, \mathbf{k}) = -j\omega\varepsilon_h A_c \frac{k_p^2}{k_h^2 - k_z^2} [E_z(\omega, \mathbf{k}) + V_{ext}A_c]. \quad (50)$$

When this expression is substituted into the Fourier-transformed equations (1)–(2), the E_z -proportional term of (50) is combined with the term $j\omega\varepsilon_h\mathbf{E}$ which results in the spatially dispersive permittivity (33), and the V_{ext} -proportional term occurs as an additional external current density

$$\mathbf{J}_{ext,V}(\omega, \mathbf{k}) = -j\omega\varepsilon_h V_{ext} \frac{k_p^2 A_c}{k_h^2 - k_z^2} \hat{\mathbf{z}}. \quad (51)$$

Therefore, applying the inverse Fourier transform, we find that

$$\mathbf{J}_{ext,V}(\omega, \mathbf{r}) = \frac{k_p^2 A_c V_{ext} e^{-jk_h|z|}}{2\eta_h} \delta(x, y) \hat{\mathbf{z}}, \quad (52)$$

where $\eta_h = \sqrt{\mu_0/\varepsilon_h}$. Thus, a lumped voltage source inserted into a wire of the unbounded uniaxial wire medium (with PEC wires, $Z_w = 0$) may be equivalently represented with a line of z -directed wave-like current (52). It is curious to note that while the lumped voltage source excitation is localized at the origin, the equivalent current density is distributed over the entire z -axis. At first sight, this may look inconsistent with causality. However, it is simple to verify that such a current is just a wave emerging from the discontinuity point at $z = 0$. Indeed, if one calculates the inverse Fourier transform of (52) with respect to time, it is found that:

$$\mathbf{J}_{ext,V}(t, \mathbf{r}) = k_p^2 \frac{A_c}{2\eta_h} \tilde{V}_{ext} \left(t - \frac{|z|}{v_h} \right) \hat{\mathbf{z}} \delta(x, y), \quad (53)$$

with $v_h = 1/\sqrt{\varepsilon_h\mu_0}$ the velocity of propagation in the host material, and $\tilde{V}_{ext}(t)$ the inverse Fourier transform of $V_{ext}(\omega)$. The above formula is manifestly consistent with causality, because the excitation at a given point z only depends on the excitation at the origin with a delay $|z|/v_h$.

V. ENERGY CONSERVATION IN THE UNIAxIAL WIRE MEDIUM AND POYNTING THEOREM

In what follows, we prove that the formulation based on the introduction of additional variables of Section II, enables formulating an energy conservation theorem and

the definition of a Poynting vector in the uniaxial wire medium.

We start with equations (1)–(4) written in time domain. The host permittivity ε_h is assumed dispersionless and lossless, and the wires are modeled by a self-impedance of the form $Z_w(\omega) = j\omega L_{\text{kin}} + R$, where the parameters L_{kin} and R are independent of frequency. For metallic wires with radius r_w standing in air and described by the Drude model with the plasma frequency ω_m and the collision frequency Γ , these parameters are $L_{\text{kin}} = 1/(\varepsilon_0 \pi r_w^2 \omega_m^2)$ and $R = \Gamma/(\varepsilon_0 \pi r_w^2 \omega_m^2)$.

Thus, in the time domain the equations (1)–(2) and (3)–(4) may be written as

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \quad (54)$$

$$\nabla \times \mathbf{H} = \varepsilon_h \frac{\partial \mathbf{E}}{\partial t} + \frac{I}{A_c} \hat{\mathbf{z}} + \mathbf{J}_{\text{ext}}, \quad (55)$$

and

$$\frac{\partial \varphi_w}{\partial z} = -(L + L_{\text{kin}}) \frac{\partial I}{\partial t} - RI + E_z + \mathcal{E}_{\text{ext}}, \quad (56)$$

$$\frac{\partial I}{\partial z} = -C \frac{\partial \varphi_w}{\partial t}, \quad (57)$$

where \mathcal{E}_{ext} is the effective EMF of the voltage sources inserted into the wires, per unit length of the wires [e.g., for a lumped source V_{ext} inserted into a wire at $\mathbf{r} = 0$, $\mathcal{E}_{\text{ext}} = V_{\text{ext}} A_c \delta(\mathbf{r})$].

Following a standard procedure, we obtain from (54)–(55):

$$\nabla \cdot [\mathbf{E} \times \mathbf{H}] = -\frac{\partial}{\partial t} \left[\frac{\varepsilon_h \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2} \right] - \mathbf{E} \cdot \mathbf{J}_{\text{ext}} - \frac{E_z I}{A_c}. \quad (58)$$

On the other hand, from (56)–(57) we have

$$\frac{\partial(\varphi_w I)}{\partial z} = -\frac{\partial}{\partial t} \left[\frac{C \varphi_w^2}{2} + \frac{L_{\text{tot}} I^2}{2} \right] - RI^2 + E_z I + \mathcal{E}_{\text{ext}} I, \quad (59)$$

where $L_{\text{tot}} = L + L_{\text{kin}}$.

Diving the last relation by A_c and adding it to (58) we obtain the conservation law

$$\nabla \cdot \mathbf{S} = -\frac{\partial W}{\partial t} - P_{\text{loss}} + P_{\text{ext}}, \quad (60)$$

where

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} + \frac{\varphi_w I}{A_c} \hat{\mathbf{z}}, \quad (61)$$

$$W = \frac{\varepsilon_h \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2} + \frac{C \varphi_w^2}{2A_c} + \frac{L_{\text{tot}} I^2}{2A_c}, \quad (62)$$

$$P_{\text{loss}} = \frac{RI^2}{A_c}, \quad (63)$$

$$P_{\text{ext}} = \frac{\mathcal{E}_{\text{ext}} I}{A_c} - \mathbf{E} \cdot \mathbf{J}_{\text{ext}}. \quad (64)$$

The vectorial quantity \mathbf{S} in (60) and (61) may be understood as the Poynting vector in the uniaxial wire medium,

and P_{ext} as the volume density of the power transferred by the external sources to the medium. In the absence of loss, i.e. when $R = 0$, the term W is univocally identified with the density of stored energy. In contrast, if loss is present, then it is generally impossible to separate the energy storage rate from the energy loss rate when a metamaterial is considered *macroscopically*.

However, if the *microstructure* of a metamaterial is known, the stored energy can be found from a consistent physical model that fully describes the processes within a unit volume of the metamaterial. Thus, if we assume that the Drude model is such a consistent model for the dynamics of the free electron plasma in metals, then (62) preserves the meaning of the stored energy density even when $R > 0$. In this case, the quantity P_{loss} has the physical meaning of an instantaneous power loss density.

Evidently, in time-harmonic regime the time-averaged Poynting vector is given by

$$\mathbf{S}_{\text{av}} = \frac{1}{2} \text{Re} \left\{ \mathbf{E} \times \mathbf{H}^* + \frac{\varphi_w I^*}{A_c} \hat{\mathbf{z}} \right\}. \quad (65)$$

It can be checked that in the lossless case ($\text{Re}\{Z_w\} = 0$), and for the case of fields with a spatial dependence of the form $e^{-j\mathbf{k} \cdot \mathbf{r}}$ with \mathbf{k} real-valued this reduces to the formula,

$$\mathbf{S}_{\text{av},l} = \frac{1}{2} \text{Re} \{ (\mathbf{E} \times \mathbf{H}^*)_l \} - \frac{\omega}{4} \mathbf{E}^* \cdot \frac{\partial \bar{\varepsilon}}{\partial k_l}(\omega, \mathbf{k}) \cdot \mathbf{E}, \quad (66)$$

with $l = x, y, z$ and $\bar{\varepsilon}$ defined as in Eq. (33), which is applicable to plane waves in general lossless spatially dispersive media.^{23,24,29} The application of the above formula to wire media has been considered in several works.^{10,30}

VI. RADIATION PATTERN IN THE PEC CASE

Next, we obtain the radiation pattern, directive gain, directivity, and radiation resistance for the case of a short vertical dipole radiating in a wire medium formed by PEC wires (Fig. 1a). We do not discuss in details the case wherein the metamaterial is excited by a lumped voltage source, because as discussed in Section III C in such a scenario the radiated field is guided along the z -axis with no decay. In particular, this implies immediately that the directivity in such a configuration is infinite.

A. Asymptotic form of the radiated fields

To begin with, we obtain the asymptotic form of the field radiated by a short vertical dipole ($V_{\text{ext}} = 0$) embedded in a wire medium formed by PEC wires when $r \rightarrow \infty$. Evidently, from the results of Section III C, unless the frequency of operation is larger than the plasma frequency of the effective medium the radiated fields will decay exponentially away from the source. Hence, in what follows we assume that $\omega > \omega_p = k_p/\sqrt{\mu_0 \varepsilon_h}$, so that $k_{ef} > 0$ in

Eq. (29). Substituting (29) into Eqs. (6)–(7), it can be easily checked that

$$\mathbf{H} \doteq -\omega k_{ef} \Phi \sin \theta \hat{\varphi}, \quad (67)$$

$$\mathbf{E} \doteq k_h^2 \frac{\Phi}{\varepsilon_h} \left[\left(1 - \frac{k_{ef}^2}{k_h^2} \right) \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta} \right], \quad (68)$$

where the symbol \doteq indicates that the identities are asymptotic ($r \rightarrow \infty$), $\Phi = p_e \frac{1}{4\pi r} e^{-jk_{ef}r}$, and $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\varphi})$ define an orthogonal reference system associated with the usual spherical coordinate system (r, θ, φ) . As seen, unlike what happens in an isotropic medium, the electric far field has a radial component. The amplitude of the electromagnetic fields varies asymptotically as $1/r$, and $E_\theta = \eta_{ef} H_\varphi$ with $\eta_{ef} = \omega \mu_0 / k_{ef}$.

Similarly, substituting (29) into Eqs. (3)–(4) and (11), it is found that the asymptotic forms of the current and additional potential are

$$I \doteq -j\omega A_c k_p^2 \Phi, \quad (69)$$

$$\varphi_w \doteq I \frac{k_{ef}}{\omega C} \cos \theta \doteq -\frac{j k_{ef}}{\varepsilon_h} \cos \theta \Phi. \quad (70)$$

Thus, from (68) and (70), we see that the time averaged Poynting vector (65) in the far-field is

$$\begin{aligned} \mathbf{S}_{av} \doteq & \frac{1}{2} \omega^3 \mu_0 |\Phi|^2 k_{ef} \sin \theta \left(\frac{k_p^2}{k_h^2} \cos \theta \hat{\theta} + \sin \theta \hat{\mathbf{r}} \right) \\ & + \frac{1}{2} \frac{k_{ef}}{\varepsilon_h} k_p^2 \omega \cos \theta |\Phi|^2 \hat{\mathbf{z}}. \end{aligned} \quad (71)$$

Straightforward calculations show that the Poynting vector only has a radial component:

$$\mathbf{S}_{av} \doteq \frac{1}{2} k_{ef} \omega^3 \mu_0 |\Phi|^2 \left(\sin^2 \theta + \frac{k_p^2}{k_h^2} \cos^2 \theta \right) \hat{\mathbf{r}}. \quad (72)$$

Hence, in part surprisingly, it is seen that a short vertical dipole embedded in a wire medium *can* radiate energy along the direction of vibration, i.e. along the z -axis! Moreover, in the limit $\omega \rightarrow \omega_p$ the radiation pattern becomes isotropic: $\mathbf{S}_{av} \approx \frac{1}{2} k_{ef} \omega^3 \mu_0 |\Phi|^2 \hat{\mathbf{r}}$. Note, however, that for $\omega = \omega_p$, we have $k_{ef} = 0$ and thus the Poynting vector vanishes in the far-field. However, slightly above ω_p the emission of radiation is certainly possible. Notice also that in the limit where $k_p \rightarrow 0$, we recover the far-field of a short vertical dipole embedded in a dielectric with permittivity ε_h .

B. The radiation intensity, directive gain, directivity, and radiation resistance

The radiation intensity of the short vertical dipole, $U = \lim_{r \rightarrow \infty} r^2 \mathbf{S}_{av}$, is given by

$$U = \frac{|p_e|^2}{32\pi^2} k_{ef} \omega^3 \mu_0 \left(\sin^2 \theta + \frac{k_p^2}{k_h^2} \cos^2 \theta \right). \quad (73)$$

Hence, the power radiated by the dipole, $P_{rad} = \int U d\Omega = 2\pi \int U \sin \theta d\theta$, is such that

$$P_{rad} = \frac{|p_e|^2}{12\pi} k_{ef} \omega^3 \mu_0 \left(1 + \frac{k_p^2}{2k_h^2} \right). \quad (74)$$

The directive gain, $g = 4\pi U / P_{rad}$, is

$$g(\theta, \varphi) = \frac{3}{2 + k_p^2/k_h^2} \left(\sin^2 \theta + \frac{k_p^2}{k_h^2} \cos^2 \theta \right). \quad (75)$$

Since $k_h \geq k_p$ it can be checked that the direction of maximal radiation is $\theta = \pi/2$. The directivity of the short vertical dipole is, thus,

$$D = \frac{3}{2 + k_p^2/k_h^2}, \quad (76)$$

which therefore increases from unity (for $k_h \approx k_p$) up to 3/2 in the limit $k_h \gg k_p$.

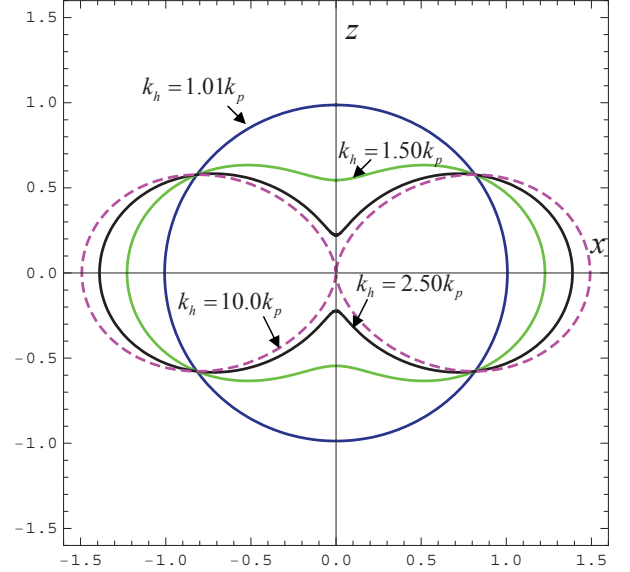


FIG. 4: (Color online) Polar plot of the directive gain of a short vertical dipole embedded in the uniaxial wire medium for different frequencies of operation ($\omega = k_h / \sqrt{\varepsilon_h \mu_0}$).

In Fig. 4 we show a polar plot of the directive gain of the short vertical dipole for different frequencies of operation, normalized to the effective plasma frequency. In agreement with the previous discussion, it can be seen that the radiation pattern becomes more directive for increasing values of the frequency, and that for $\omega \approx \omega_p$ the radiator resembles an isotropic radiator.

To conclude, we note that if the dipole is fed by a current I_0 and has infinitesimal height dl , then the corresponding dipole moment is such that $\omega |p_e| = |I_0| dl$. Thus, it follows that the radiation resistance ($R_{rad} =$

$2P_{rad}/|I_0|^2$) of such an elementary source is given by

$$\begin{aligned} R_{rad} &= \frac{(dl)^2}{6\pi} k_{ef} \omega \mu_0 \left(1 + \frac{k_p^2}{2k_h^2} \right) \\ &= \eta_h \frac{(dl)^2}{6\pi} k_{ef} k_h \left(1 + \frac{k_p^2}{2k_h^2} \right), \end{aligned} \quad (77)$$

where $\eta_h = \sqrt{\mu_0/\epsilon_h}$ is the impedance of the host material.

VII. CONCLUSION

In this work we have studied the radiation of two types of elementary sources embedded in a uniaxial wire

medium and derived a general energy conservation theorem. The main challenge of the radiation problem is related to the metamaterial being spatially dispersive. We have shown that the radiation problem can be solved by considering either a nonlocal dielectric function framework or, alternatively, a local model framework based on the introduction of additional variables. However, only the latter approach enables considering stratified media and calculating quantities such as the Poynting vector or the directive gain. It was shown that the emission of radiation by a short dipole in a wire medium has several anomalous features, such as a uniform directive gain near the effective plasma frequency. On the other hand, the radiation by a lumped voltage generator results in a non-diffractive beam that is localized in the vicinity of the z -axis, and corresponds to an infinite directivity.

-
- * Electronic address: mario.silveirinha@co.it.pt
† Electronic address: stas@co.it.pt
- ¹ J. Brown, Proc. IEE **100**, 51 (1953).
 - ² W. Rotman, IRE Trans. Antennas Propag. **10**, 82 (1962).
 - ³ J. B. Pendry, A. J. Holden, W. J. Stewart, and I. Youngs, Phys. Rev. Lett. **76**, 4773 (1996).
 - ⁴ S. I. Maslovski, S. A. Tretyakov, and P. A. Belov, Microw. Opt. Techn. Lett. **35**, 47 (2002).
 - ⁵ A. L. Pokrovsky and A. L. Efros, Phys. Rev. B **65**, 045110 (2002).
 - ⁶ P. A. Belov, R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, Phys. Rev. B **67**, 113103 (2003).
 - ⁷ M. G. Silveirinha and C. A. Fernandes, IEEE Trans. on Microwave Theory and Tech. **53**, 1418 (2005).
 - ⁸ C. R. Simovski and P. A. Belov, Phys. Rev. E **70**, 046616 (2004).
 - ⁹ I. S. Nefedov, A. J. Viitanen, and S. A. Tretyakov, Phys. Rev. E **71**, 046612 (2005).
 - ¹⁰ I. S. Nefedov, A. J. Viitanen, and S. A. Tretyakov, Phys. Rev. B **72**, 245113 (2005).
 - ¹¹ M. G. Silveirinha, IEEE Trans. Antennas Propagat **54**, 1766 (2006).
 - ¹² M. G. Silveirinha, Phys. Rev. E **73**, 046612 (2006).
 - ¹³ M. G. Silveirinha, *Nonlocal Homogenization Theory of Structured Materials, chapter in Theory and Phenomena of Artificial Materials vol. 1 (edited by F. Capolino)* (CRC, 2009).
 - ¹⁴ M. G. Silveirinha and C. A. Fernandes, Phys. Rev. B **78**, 033108 (2008).
 - ¹⁵ O. Luukkonen, M. G. Silveirinha, A. B. Yakovlev, C. R. Simovski, I. S. Nefedov, and S. A. Tretyakov, IEEE Trans. Microwave Theory Tech. **57**, 2692 (2009).
 - ¹⁶ A. B. Yakovlev, M. G. Silveirinha, O. Luukkonen, C. R. Simovski, I. S. Nefedov, and S. A. Tretyakov, IEEE Trans. Microwave Theory Tech. **57**, 2700 (2009).
 - ¹⁷ S. I. Maslovski and M. G. Silveirinha, Phys. Rev. B **80**, 245101 (2009).
 - ¹⁸ I. S. Nefedov, Phys. Rev. B **82**, 155423 (2010).
 - ¹⁹ P. Ikonen, M. Karkkainen, C. Simovski, P. Belov, and S. Tretyakov, IEE Proc.-Microw. Antennas Propag. **153**, 163 (2006).
 - ²⁰ G. Lovat, P. Burghignoli, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Antennas Propagat. **54**, 1017 (2006).
 - ²¹ P. Burghignoli, G. Lovat, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Antennas Propagat. **56**, 1329 (2008).
 - ²² P. Burghignoli, G. Lovat, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Microwave Theory Tech. **58**, 1112 (2008).
 - ²³ V. M. Agranovich and V. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons* (Wiley-Interscience N.Y., 1966).
 - ²⁴ J. T. Costa, M. G. Silveirinha, and A. Alu, Phys. Rev. B **83**, 165120 (2011).
 - ²⁵ S. I. Maslovski, T. Morgado, M. G. Silveirinha, C. S. R. Kaipa, and A. B. Yakovlev, New J. Phys. **12**, 113047 (2010).
 - ²⁶ M. G. Silveirinha, P. A. Belov, and C. R. Simovski, Phys. Rev. B **75**, 035108 (2007).
 - ²⁷ K. Sakoda, *Optical Properties of Photonic Crystals* (Springer-Verlag (Berlin, Heidelberg), 2001).
 - ²⁸ P. A. Belov, Y. Hao, and S. Sudhakaran, Phys. Rev. B. **73**, 033108 (2006).
 - ²⁹ M. G. Silveirinha, Phys. Rev. B **80**, 235120 (2009).
 - ³⁰ M. G. Silveirinha, New J. Phys. **11**, 113016 (2009).